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Direct observation of normal modes in coupled oscillators

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We propose a simple and inexpensive method to directly observe each normal mode of a system of coupled oscillators, as well as to measure its corresponding frequency, without performing Fourier analysis or using expensive apparatus. The method consists of applying a frequency dependent force to the system and using the resonance to excite each mode separately. The frequency of the excited mode is determined by measuring the resonance frequency of the system. We found that the measured normal mode frequencies of coupled oscillators exhibiting two and three normal modes are in very good agreement with the theoretical estimates. The method is suitable for undergraduate students with an elementary knowledge of differential equations. © *2003 American Association of Physics Teachers.*
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I. INTRODUCTION

A variety of systems in physics, chemistry, and engineering can be described by a set of coupled oscillators each having a well-defined frequency of vibration. A solid, for instance, is composed of a large number of atoms (or molecules) arranged in a lattice. These lattice points represent the equilibrium position of the atoms undergoing thermal vibrations. Because of their interactions, the motion of a given atom affects its neighbors. Due to the large number of atoms in a solid, an accurate description of their individual motion would present insurmountable difficulties. However, many physical properties of solids are not determined by the individual behavior of the constituting atoms, but instead by their collective behavior. In many applications each atom may be represented by a spring-mass system having a well-defined frequency of vibration. Due to the coupling between the atoms, the motion of each oscillator shows a complicated pattern of motion, and vibrates in a multifrequency fashion. In a system of coupled oscillators described by N degrees of freedom, there are N unique patterns of vibration in which all masses oscillate at the same frequency with fixed amplitudes. Such patterns of vibrations are called normal modes. The importance of these modes is that the general motion of any mass of the system consists of a linear combination of the individual normal modes.

Several papers describe methods to study normal modes in coupled oscillators.[1–6] These methods are quite accurate, and use computer assisted Fourier analysis. However, due to their relative mathematical and experimental sophistication, they can only be used in an advanced laboratory for upper-division undergraduates. More recent work[7] also investigates the normal modes of coupled oscillators using video analysis. Although less mathematically complex, it requires the use of a nonlinear fitting procedure to determine the normal mode frequencies as well as relatively expensive video equipment and commercial software packages. We propose a method that is inexpensive, mathematically simple, and allows direct observation of the normal modes in coupled oscillators without requiring the use of fast Fourier transforms or video analysis. The method consists of observing the behavior of coupled oscillators in the presence of an external periodic force. A normal mode of the system is found by systematically changing the frequency of the driving force until the system exhibits resonance. This method is therefore suitable for undergraduates with an elementary knowledge of differential equations.

II. THEORY

Consider the system shown in Fig. 1 consisting of two masses m_1 and m_2 connected to each other and two other points by three springs with elastic constants k_1, k_2, and k_3. The system is vertically aligned. The top spring is connected to a fixed point and the bottom spring is allowed to move vertically about an initially fixed point. We consider one-dimensional motion along the line connecting the masses. The motion of the system is characterized by two degrees of freedom, one for each mass. We define the coordinates, x_1 and x_2, as the downward displacement from equilibrium of m_1 and m_2, respectively, after both masses are connected to the springs. If we apply Newton's law to each mass, we find the following equations of motion:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1, \tag{1a}$$

$$m_2\ddot{x}_2 = -k_2x_2 - k_2(x_2 - x_1) - c_2\dot{x}_2 + k_3D\sin(\omega t). \tag{1b}$$

The absence of the gravitational force on each of the masses in Eq. (1) is due to the appropriate choice of the origin of coordinates, which cancels the effects of gravity. The parameters c_1 and c_2, are the coefficients of the velocity dependent damping force. D and ω, respectively, are the amplitude and frequency of the displacement applied to the bottom spring. For simplicity, we shall work with a system of identical masses m, and identical spring constants k. From symmetry arguments we also take $c_1 = c_2 = c$. The equations of motion now read

$$m\ddot{x}_1 = -2kx_1 + kx_2 - c\dot{x}_1, \tag{2a}$$

$$m\ddot{x}_2 = -2kx_2 + kx_1 - c\dot{x}_2 + F_0\sin(\omega t), \tag{2b}$$

where $F_0 \equiv kD$ is the amplitude of the external periodic force.

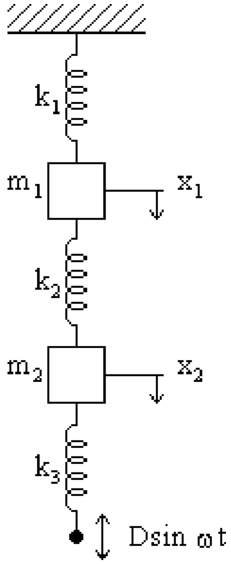

Fig. 1. Schematics of the experimental arrangement for a system of coupled oscillators consisting of two masses and three springs. One of the ends of the bottom spring is connected to a wave driver and allowed to oscillate in the vertical direction.

Equations (2a) and (2b) are coupled, that is, the motion of one of the masses will affect the subsequent motion of the other. It is possible to find a new set of coordinates that uncouples the motion, such that each new coordinate oscillates independently with a well-defined frequency. For the system discussed here, the new coordinates are $q_1 = x_1 + x_2$ and $q_2 = x_2 - x_1$. By adding and subtracting Eqs. (2a) and (2b), we obtain

$$m\ddot{q}_1 + c\dot{q}_1 + kq_1 = F_0 \sin(\omega t), \tag{3a}$$

$$m\ddot{q}_2 + c\dot{q}_2 + 3kq_2 = F_0 \sin(\omega t). \tag{3b}$$

Each of the two independent equations corresponds to the motion of a damped harmonic oscillator in the presence of a periodic external force. The first oscillator has an angular frequency given by $\omega_1 = \sqrt{k/m - \gamma^2}$, and the second by $\omega_2 = \sqrt{3k/m - \gamma^2}$, where $\gamma = c/2m$. The general solution to either Eq. (3a) or (3b) consists of the sum of the two parts. The first part, is the transient solution corresponding to the solution of Eq. (3) in the absence of a driving force. This solution is short lived, and plays no role in the method discussed here. The second part, q_i^s, corresponds to the steady-state solution in the presence of the driving force. This solution reads as

$$q_i^s(t) = A_i^s \sin(\omega t - \phi_i), \tag{4}$$

where

$$A_i^s = \frac{F_0/m}{\sqrt{(\omega_i^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \tag{5}$$

and $\phi_i = \tan^{-1}[2\gamma\omega/(\omega_i^2 - \omega^2)]$. Equation (4) represents the motion of a harmonic oscillator whose phase and amplitude depend on the frequency of the external force and the strength of the damping force. The maximum amplitude occurs at the resonant frequency $\omega = \omega_r$, where $\omega_r = \sqrt{\omega_i^2 - \gamma^2}$ (see, for example, Ref. 8).

After the transient term is no longer relevant, the positions of the masses are given by

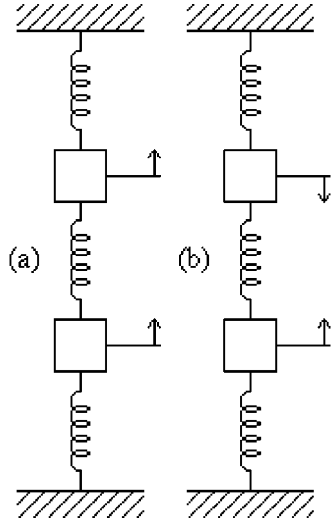

Fig. 2. Modes of vibration for a system of coupled oscillators with two masses and three springs. (a) The symmetric mode has the lowest frequency and the motion of the masses is in phase. (b) In the antisymmetric mode, the motion is out of phase.

$$x_1(t) = A_1 \sin(\omega t - \phi_1) - A_2 \sin(\omega t - \phi_2), \tag{6a}$$

$$x_2(t) = A_1 \sin(\omega t - \phi_1) + A_2 \sin(\omega t - \phi_2), \tag{6b}$$

where $x_1 = (q_1^s - q_2^s)/2$, $x_2 = (q_1^s + q_2^s)/2$, and $A_i = A_i^s/2$. Equation (6) shows that the motion of each mass consists of a linear combination of two forced damped harmonic oscillators with the same frequency but different amplitudes and phases. Note that the motion of each mass no longer depends on the initial conditions. To experimentally observe a given normal mode, we adjust the external frequency so that it resonates with the desired normal mode. When that occurs, the amplitude of that mode reaches a maximum. Because the oscillators are in the lightly damped regime, the amplitudes of the other modes will be negligibly small. Notice that the resonant frequency is not the same as the normal mode frequency, but they are approximately equal in the lightly damped regime. A similar procedure has been proposed for finding the resonant frequencies of coupled oscillators using computer simulations.[9]

Although our discussion so far has been restricted to the motion of two oscillators, the method may also be applied to any number of oscillators. We have simplified the discussion by considering identical oscillators, but the method also works if the oscillators are not identical. The theoretical expressions for the normal mode frequencies for different masses and springs of unequal elastic constants are rather cumbersome and are found in Ref. 1.

III. EXPERIMENTAL PROCEDURE

The experimental setup is indicated in Fig. 1. The end of the spring k_3, is connected to a wave driver. The wave driver provides the frequency-dependent external force applied to the system. We used identical springs, each with $k = 16.5 \pm 0.1$ N/m, and masses of 50 ± 1 g each. The wave driver was made from a loudspeaker purchased from a local electronics store.[10] To transform the loudspeaker into a wave driver, we use a plastic disk, 3 mm thick and 9.5 cm in diameter. We glued it to the cone of the loudspeaker, perpendicularly to its symmetry axis. In this way, as the cone vi-

Table I. Normal modes frequencies for a system of coupled oscillators with two masses and three springs (see Fig. 2). Both the calculated and measured values are shown. The expressions for the theoretical values of the frequencies are in the text.

Frequency (Hz)	Calculated	Measured
f_1	2.89 ± 0.04	2.8 ± 0.1
f_2	5.01 ± 0.07	5.0 ± 0.1

Table II. Measured and calculated normal modes frequencies for a system of coupled oscillators with three masses and four springs (see Fig. 3). The expressions for the theoretical values of the frequencies are in the text.

Frequency (Hz)	Calculated	Measured
f_1	2.21 ± 0.03	2.1 ± 0.1
f_2	4.09 ± 0.06	4.1 ± 0.1
f_2	5.34 ± 0.07	5.3 ± 0.1

brates so does the disk. To the center of the disk we screwed in a threaded hook to which the spring was attached. The other end of the spring was attached to a mass. The wave driver was connected to a frequency generator via a low-power amplifier. The amplifier was built from a kit.[11] The entire setup cost less than $25, not including the frequency generator, which is presumably available in any introductory physics laboratory.[12]

To observe the normal modes, we start by driving the system with a low frequency (about 1.0 Hz) and low amplitude. The motion of the system will initially be irregular. For a fixed driving amplitude, we slowly and systematically increase the driving frequency until the motion of the oscillators becomes automatically synchronized. The masses will oscillate with the largest amplitude when the driving frequency is equal to one of the normal modes of the system. By further increasing the frequency of the generator and following the same procedure as before, the remaining normal modes are easily found.

IV. RESULTS

Consider first the case of two identical masses connected by three identical springs, as depicted in Fig. 2. There are

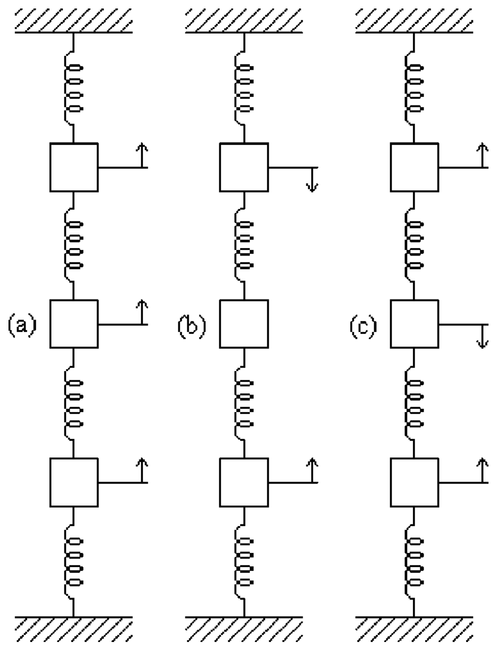

Fig. 3. Modes of vibration for a system of coupled oscillators with three masses and four springs. (a) Normal mode with the lowest frequency. The motion of the three masses is in phase. (b) Normal mode with intermediate frequency. The central mass is at rest and the motion of the other masses is out of phase. (c) Highest frequency normal mode. The motion of the outer masses is in phase while the motion of center mass is out of phase with the others.

two normal modes in the system. Because the damping coefficient is very small in our experiments, in what follows we will disregard the damping coefficient in the expressions for the normal mode frequencies. The one with the lowest frequency is the symmetric mode [Fig. 2(a)]. In this mode the masses are moving in the same direction at all times with frequency $f_1=\omega_1/2\pi=(1/2\pi)\sqrt{k/m}$. The other mode [Fig. 2(b)] is the antisymmetric mode, where the masses are moving in the opposite direction at all times with the frequency $f_2=\omega_2/2\pi=(1/2\pi)\sqrt{3k/m}$. By following the experimental procedure previously described, we find $f_1=2.8\pm0.1$ Hz and $f_2=5.0\pm0.1$ Hz. For comparison, in Table I we present these results together with the expected theoretical values. By repeating the theoretical analysis for a system having three identical masses and four identical springs, we find three normal modes, schematically shown in Fig. 3. As expected, the mode with the lowest frequency is the one with highest symmetry [Fig. 3(a)]. This mode corresponds to the motion of all masses in phase with frequency $f_1=(1/2\pi)\sqrt{(2-\sqrt{2})k/m}$, and the central mass having an amplitude greater than the masses on either side, by a factor of $\sqrt{2}$. In the second mode [Fig. 3(b)] the central mass remains stationary, while the masses on either side oscillate with the same amplitude in opposite directions and frequency $f_2=(1/2\pi)\sqrt{2k/m}$. The mode with the highest frequency [Fig. 3(c)] has the lowest symmetry. In this mode the masses on either side oscillate in the same direction, while the central mass is out of phase with them. They move with a frequency given by $f_3=(1/2\pi)\sqrt{(2+\sqrt{2})k/m}$. As for the mode of lowest frequency, the central mass oscillates with an amplitude greater than the other masses by a factor of $\sqrt{2}$. The measured values of the vibrational modes are summarized together with the theoretical estimates in Table II. In both experiments we estimated the error in the measurement of any frequency to be about 0.1 Hz. This uncertainty is due mainly to the inaccuracy in reading the frequency from the dial of the function generator.

To conclude, we have presented a method that allows direct observation as well as determination of the normal modes of vibration of a system of coupled oscillators. The advantage of this method is that it does not require the use of fast Fourier transforms or video analysis to measure the normal mode frequencies. Also, each normal mode of the system can actually be seen in the experiment. The low cost of the apparatus and simplicity of the method make it suitable for students at the beginning undergraduate level.

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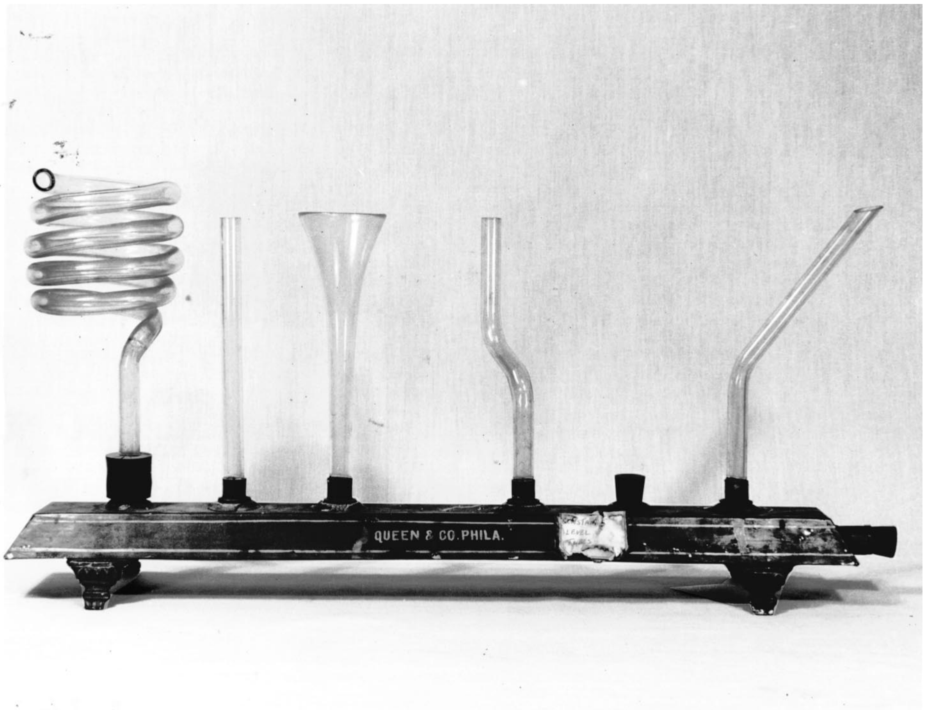
[10]In our experiment we used a 8Ω loudspeaker from Radio Shack, catalog No. 40-1345A.
[11]We used a 7-watt mono amplifier kit from Jameco, part No. 117612.
[12]For the results presented in this paper we used the LG-Precision Function Generator FG-8002. We have also used the BK-Precision Function Generator 3020, which gave similar results. Both function generators are low cost and have a calibrated knob-dial frequency selector.



Liquid Level Device. The water in each of the tubes in the liquid level device is subject to the same atmospheric pressure at the upper surface. If the base is level, the pressure at the bottom of each tube must be the same. Therefore, the depth of the water in each tube is independent of its shape, and is the same. The demonstration is still done today, although not with apparatus as attractive as the typical nineteenth century device shown here. This was made by the Philadelphia firm of James W. Queen & Co., and was listed in the 1881 catalogue at $3.50. The missing tube was wide at the bottom and tapered to a narrow top. This example is at St. Mary's College in Notre Dame, Indiana. (Photograph and notes by Thomas B. Greenslade, Jr., Kenyon College)